Chapter 3

Fuzzy Averaging for Forecasting

Forecasting[1] provides the basis for any production activity. The ability to predict and estimate future events requires the study of imprecise data information coming from a rapidly changing environment, a task for which fuzzy logic is better suited to deal with than classical methods. Analysis of complex situations needs the efforts and opinions of many experts. The experts opinions, almost never identical, are either more or less close or more or less conflicting. They have to be combined or aggregated in order to produce one conclusion. In this chapter the methodology of fuzzy averaging is introduced. It is used as a major tool for aggregation in various forecasting models (fuzzy Delphi, project management, forecasting demand). In Chapter 4 fuzzy averaging is applied to decision making.

3.1 Statistical Average

One of the most important concepts in statistics is the *average* or *mean* of n measurements, readings, or estimates expressed by real numbers $r_1, \ldots, r_n$. It is defined by

$$r_{ave} = \frac{r_1 + \cdots + r_n}{n} = \frac{\sum_{i=1}^{n} r_i}{n}; \tag{3.1}$$

the measurements are considered of equal importance. The average which is typical or representative of n measurements is also known as a *measure of central tendency.*

If the measurements $r_1, \ldots, r_n$ have different importance expressed by the real numbers $\lambda_1, \ldots, \lambda_n$, correspondingly, then the concept of *weighted average* or *weighted mean* is introduced by the formula

$$r_{ave}^{w} = \frac{\lambda_1 r_1 + \cdots + \lambda_n r_n}{\lambda_1 + \cdots + \lambda_n} = w_1 r_1 + \cdots + w_n r_n = \sum_{i=1}^{n} w_i r_i. \tag{3.2}$$

Here w_i called *weights* are given by

$$w_i = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}, \quad i = 1, \ldots, n, \quad w_1 + \cdots + w_n = \sum_{i=1}^{n} w_i = 1. \tag{3.3}$$

The weights reflect the relative importance or strength of the measurements r_i.

The concept of average, we may call it *crisp average*, can be generalized by substituting fuzzy numbers for the real numbers r_i in formulas (3.1) and (3.2). For that purpose arithmetic operations with fuzzy numbers have to be performed, which in general requires complicated computations. Here we restrict the generalization procedure to triangular and trapezoidal numbers. They are used very often in applications and besides, it is easy to perform arithmetic operations with them; this is demonstrated in the next section.[2]

3.2 Arithmetic Operations with Triangular and Trapezoidal Numbers

Addition of triangular numbers

It can be proved that the sum of two triangular numbers $\mathbf{A}_1 = (a_1^{(1)}, a_M^{(1)}, a_2^{(1)})$ and $\mathbf{A}_2 = (a_1^{(2)}, a_M^{(2)}, a_2^{(2)})$, is also a triangular number,

$$\begin{aligned} \mathbf{A}_1 + \mathbf{A}_2 &= (a_1^{(1)}, a_M^{(1)}, a_2^{(1)}) + (a_1^{(2)}, a_M^{(2)}, a_2^{(2)}) \\ &= (a_1^{(1)} + a_1^{(2)}, a_M^{(1)} + a_M^{(2)}, a_2^{(1)} + a_2^{(2)}). \end{aligned} \tag{3.4}$$

This summation formula can be extended for n triangular numbers. Also it can be applied for left and right triangular numbers (Section 1.5). For instance:

$$\begin{aligned}\mathbf{A}_1^r + \mathbf{A}_2 &= (a_M^{(1)}, a_M^{(1)}, a_2^{(1)}) + (a_1^{(2)}, a_M^{(2)}, a_2^{(2)}) \\ &= (a_M^{(1)} + a_1^{(2)}, a_M^{(1)} + a_M^{(2)}, a_2^{(1)} + a_2^{(2)}),\end{aligned}$$

$$\begin{aligned}\mathbf{A}_1^l + \mathbf{A}_2^l &= (a_1^{(1)}, a_M^{(1)}, a_M^{(1)}) + (a_1^{(2)}, a_M^{(2)}, a_M^{(2)}) \\ &= (a_1^{(1)} + a_1^{(2)}, a_M^{(1)} + a_M^{(2)}, a_M^{(1)} + a_M^{(2)}).\end{aligned}$$

Example 3.1

The sum of the triangular numbers

$$\mathbf{A}_1 = (-5, -2, 1), \qquad \mathbf{A}_2 = (-3, 4, 12),$$

according to (3.4) is the triangular number

$$\mathbf{A}_1 + \mathbf{A}_2 = (-5 + (-3), -2 + 4, 1 + 12) = (-8, 2, 13)$$

shown on Fig. 3.1.

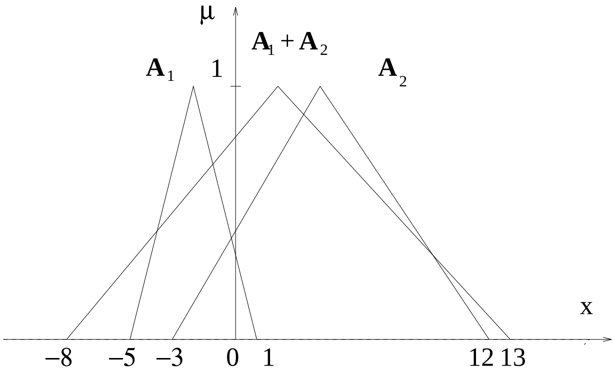

Fig. 3.1. Sum of two triangular numbers.

Figure 3.1 can be interpreted as follows. If $\mathbf{A}_1$ describes real numbers close to -2 and $\mathbf{A}_2$ describes real numbers close to 4, then $\mathbf{A}_1 + \mathbf{A}_2$ represents real numbers close to $-2 + 4 = 2$.

□

Example 3.2

Now let us find the sum of three triangular numbers:

$$\mathbf{A}_1^r = (0, 0, 2), \quad \mathbf{A}_2 = (1, 3, 4), \quad \mathbf{A}_3^l = (3, 6, 6);$$

$\mathbf{A}_1^r$ and $\mathbf{A}_3^l$ are right and left triangular numbers. The extended formula (3.4) gives (see Fig. 3.2)

$$\mathbf{A}_1^r + \mathbf{A}_2 + \mathbf{A}_3^l = (0 + 1 + 3, 0 + 3 + 6, 2 + 4 + 6) = (4, 9, 12).$$

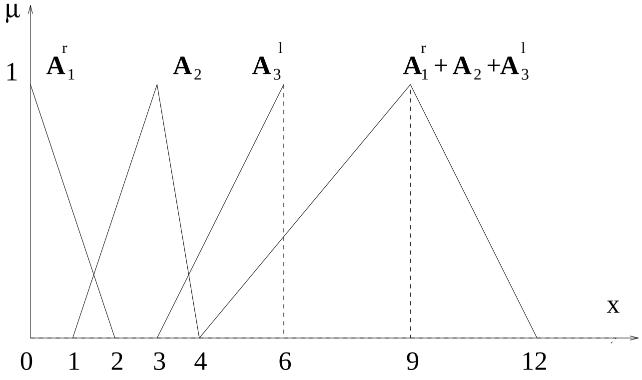

Fig. 3.2. Sum of $\mathbf{A}_1^r$, $\mathbf{A}_2$, and $\mathbf{A}_3^l$.

□

Multiplication of a triangular number by a real number

The product of a triangular number $\mathbf{A}$ with a real number r is also a triangular number,

$$\mathbf{A}r = r\mathbf{A} = r(a_1, a_M, a_2) = (ra_1, ra_M, ra_2). \tag{3.5}$$

Division of a triangular number by a real number

This operations is defined as multiplication of **A** by $\frac{1}{r}$ provided that $r \neq 0$. Hence (3.5) gives

$$\frac{\mathbf{A}}{r} = \frac{1}{r}(a_1, a_M, a_2) = (\frac{a_1}{r}, \frac{a_M}{r}, \frac{a_2}{r}). \tag{3.6}$$

Example 3.3

(a) The product of $\mathbf{A} = (2, 4, 5)$ by 2 according to (3.5) is (see Fig. 3.3)

$$2\mathbf{A} = 2(2, 4, 5) = (4, 8, 10).$$

(b) The division of $\mathbf{A} = (2, 4, 5)$ by 2 using (3.6) produces (Fig. 3.3)

$$\frac{\mathbf{A}}{2} = \frac{1}{2}(2, 4, 5) = (1, 2, 2.5).$$

(c) Also

$$\frac{2\mathbf{A}}{2} = \frac{(4, 8, 10)}{2} = \mathbf{A}, \quad 2(\frac{\mathbf{A}}{2}) = 2(1, 2, 2.5) = \mathbf{A}.$$

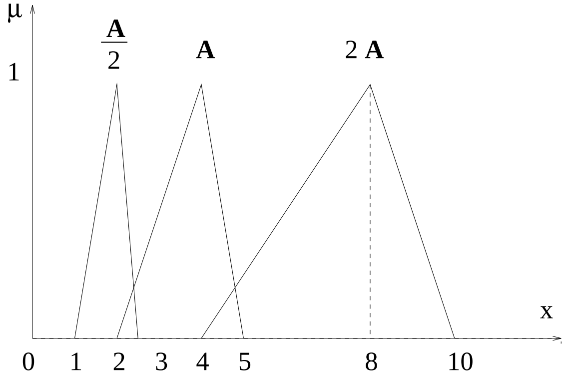

Fig. 3.3. Triangular number $\mathbf{A} = (2, 4, 5)$; product $2\mathbf{A}$; quotient $\frac{\mathbf{A}}{2}$.

□

Operations with trapezoidal numbers can be performed similarly to those with triangular numbers.

Addition of trapezoidal numbers

The sum of the trapezoidal numbers $\mathbf{A}_1 = (a_1^{(1)}, b_1^{(1)}, b_2^{(1)}, a_2^{(1)})$ and $\mathbf{A}_2 = (a_1^{(2)}, b_1^{(2)}, b_2^{(2)}, a_2^{(2)})$ is also a trapezoidal number,

$$\begin{aligned}\mathbf{A}_1 + \mathbf{A}_2 &= (a_1^{(1)}, b_1^{(1)}, b_2^{(1)}, a_2^{(1)}) + (a_1^{(2)}, b_1^{(2)}, b_2^{(2)}, a_2^{(2)}) \\ &= (a_1^{(1)} + a_1^{(2)}, b_1^{(1)} + b_1^{(2)}, b_2^{(1)} + b_2^{(2)}, a_2^{(1)} + a_2^{(2)}). \quad (3.7)\end{aligned}$$

Formula (3.7) can be generalized for n trapezoidal numbers and also for left and right trapezoidal numbers.

Multiplication of a trapezoidal number by a real number

$$\mathbf{A}r = r\mathbf{A} = (ra_1, rb_1, rb_2, ra_2). \quad (3.8)$$

Division of a trapezoidal number by a real number

$$\frac{\mathbf{A}}{r} = \frac{1}{r}\mathbf{A} = (\frac{a_1}{r}, \frac{b_1}{r}, \frac{b_2}{r}, \frac{a_2}{r}), \quad r \neq 0. \quad (3.9)$$

Sum of triangular and trapezoidal numbers

Consider the triangular number $\mathbf{A}_1 = (a_1^{(1)}, a_M^{(1)}, a_2^{(1)})$ which can be presented as a trapezoidal number $(a_1^{(1)}, a_M^{(1)}, a_M^{(1)}, a_2^{(1)})$ and the trapezoidal number $\mathbf{A}_2 = (a_1^{(2)}, b_1^{(2)}, b_2^{(2)}, a_2^{(2)})$. Using (3.7) gives

$$\begin{aligned}\mathbf{A}_1 + \mathbf{A}_2 &= (a_1^{(1)}, a_M^{(1)}, a_M^{(1)}, a_2^{(1)}) + (a_1^{(2)}, b_1^{(2)}, b_2^{(2)}, a_2^{(2)}) \\ &= (a_1^{(1)} + a_1^{(2)}, a_M^{(1)} + b_1^{(2)}, a_M^{(1)} + b_2^{(2)}, a_2^{(1)} + a_2^{(2)}). \quad (3.10)\end{aligned}$$

3.3 Fuzzy Averaging

Triangular average formula

Consider n triangular numbers $\mathbf{A}_i = (a_1^{(i)}, a_M^{(i)}, a_2^{(i)}), i = 1, \ldots, n$. Using addition of triangular numbers and division by a real number (see (3.4) and (3.6)) gives the *triangular average* (*mean*) $\mathbf{A}_{ave}$,

$$\begin{aligned}
\mathbf{A}_{ave} &= \frac{\mathbf{A}_1 + \cdots + \mathbf{A}_n}{n} \\
&= \frac{(a_1^{(1)}, a_M^{(1)}, a_2^{(1)}) + \cdots + (a_1^{(n)}, a_M^{(n)}, a_2^{(n)})}{n} \\
&= \frac{(\sum_{i=1}^n a_1^{(i)}, \sum_{i=1}^n a_M^{(i)}, \sum_{i=1}^n a_2^{(i)})}{n},
\end{aligned}$$

which is a triangular number,

$$\mathbf{A}_{ave} = (m_1, m_M, m_2) = (\frac{1}{n}\sum_{i=1}^n a_1^{(i)}, \frac{1}{n}\sum_{i=1}^n a_1^{(i)}, \frac{1}{n}\sum_{i=1}^n a_2^{(i)}). \qquad (3.11)$$

Example 3.4

(a) The triangular numbers $\mathbf{A}_1$ and $\mathbf{A}_2$ in Example 3.1 have average

$$\mathbf{A}_{ave} = \frac{\mathbf{A}_1 + \mathbf{A}_2}{2} = \frac{(-8, 2, 13)}{2} = (-4, 1, 6.5).$$

(b) The triangular numbers $\mathbf{A}_1^r, \mathbf{A}_2$, and $\mathbf{A}_3^l$ in Example 3.2 have average

$$\mathbf{A}_{ave} = \frac{\mathbf{A}_1^r + \mathbf{A}_2 + \mathbf{A}_3^l}{3} = \frac{(4, 9, 12)}{3} = (1.33, 3, 4).$$

□

Weighted triangular average formula

If the real numbers λ_i represent the importance of $\mathbf{A}_i = (a_1^{(i)}, a_M^{(i)}, a_2^{(i)})$, $i = 1, \ldots, n$, then following (3.2), using (3.3), and similarly to (3.11) we obtain the *weighted triangular average* (*mean*),

$$\begin{aligned}
\mathbf{A}_{ave}^w &= \frac{\lambda_1 \mathbf{A}_1 + \cdots + \lambda_n \mathbf{A}_n}{\lambda_1 + \cdots + \lambda_n} \\
&= w_1(a_1^{(1)}, a_M^{(1)}, a_2^{(1)}) + \cdots + w_n(a_1^{(n)}, a_M^{(n)}, a_2^{(n)}) \\
&= (w_1 a_1^{(1)}, w_1 a_M^{(1)}, w_1 a_2^{(1)}) + \cdots + (w_n a_1^{(n)}, w_n a_M^{(n)}, w_2^{(n)}) \\
&= (w_1 a_1^{(1)} + \cdots + w_n a_1^{(n)}, w_1 a_M^{(1)} + \cdots + w_n a_M^{(n)}, \\
&\quad w_1 a_2^{(1)} + \cdots + w_n a_2^{(n)}),
\end{aligned}$$

which can be written as

$$\mathbf{A}^w_{ave} = (m^w_1, m^w_M, m^w_2) = (\sum_{i=1}^n w_i a_1^{(i)}, \sum_{i=1}^n w_i a_M^{(i)}, \sum_{i=1}^n w_i a_2^{(i)}). \qquad (3.12)$$

Average formulas for trapezoidal numbers which can be derived similarly to (3.11) and (3.12) are presented below.

Trapezoidal average formula

If $\mathbf{A}_i = (a_1^{(i)}, b_1^{(i)}, b_2^{(i)}, a_2^{(i)}), i = 1, \dots, n$, are trapezoidal numbers, then

$$\begin{aligned} \mathbf{A}_{ave} &= (m_1, m_{M_1}, m_{M_2}, m_2) \\ &= \frac{(a_1^{(1)}, b_1^{(1)}, b_2^{(1)}, a_2^{(1)}) + \cdots + (a_1^{(n)}, b_1^{(n)}, b_2^{(n)}, a_2^{(n)})}{n} \\ &= \frac{(\sum_{i=1}^n a_1^{(i)}, \sum_{i=1}^n b_1^{(i)}, \sum_{i=1}^n b_2^{(i)}, \sum_{i=1}^n a_2^{(i)})}{n}. \qquad (3.13) \end{aligned}$$

Weighted trapezoidal average formula

$$\begin{aligned} \mathbf{A}^w_{ave} &= (m^w_1, m^w_{M_1}, m^w_{M_2}, m^w_2) \\ &= w_1(a_1^{(1)}, b_1^{(1)}, b_2^{(1)}, a_2^{(1)}) + \cdots + w_n(a_1^{(n)}, b_1^{(n)}, b_2^{(n)}, a_2^{(n)}) \\ &= (\sum_{i=1}^n w_i a_1^{(i)}, \sum_{i=1}^n w_i b_1^{(i)}, \sum_{i=1}^n w_i b_2^{(i)}, \sum_{i=1}^n w_i a_2^{(i)}). \qquad (3.14) \end{aligned}$$

The triangular and trapezoidal average and weighted average formulas (3.11)–(3.14) produce a result which can be interpreted as follows. It is a *conclusion* or *aggregation* of all combined meanings expressed by triangular and trapezoidal numbers $\mathbf{A}_1, \dots, \mathbf{A}_n$ considered either of equal importance or of different importance expressed by weights w_i.

Based on the arithmetic operations in Section 3.2, we can state that:

1) Formulas (3.11)–(3.14) remain valid when some of $\mathbf{A}_i$ are left or right triangular or trapezoidal numbers.

2) Formulas (3.13) and (3.14) for trapezoidal numbers remain valid when some $\mathbf{A}_i$ are triangular numbers since they can be expressed in the form of trapezoidal numbers.

The process of averaging presented here is a cross section of classical statistics and fuzzy sets theory; it belongs to a new branch of science—*fuzzy statistics.*

Defuzzification of fuzzy average

The aggregation defined by a triangular or trapezoidal average number ((3.11)–(3.14)) very often has to be expressed by a crisp value which represent best the corresponding average. This operation is called *defuzzification.*

First consider the defuzzification of $\mathbf{A}_{ave} = (m_1, m_M, m_2)$ given in (3.11). It looks plausible to select for that purpose the value m_M in the supporting interval $[m_1, m_2]$ of $\mathbf{A}_{ave}$; m_M has the highest degree (one) of membership in $\mathbf{A}_{ave}$. In other words, $\mathbf{A}_{ave}$ attains its maximum at

$$x_{\max} = m_M \tag{3.15}$$

which we call *maximizing value.*

However the operation defuzzification can not be defined uniquely. Here we present three options for defuzzifying $\mathbf{A}_{ave} = (m_1, m_M, m_2)$ which are essentially statistical average formulas:

$$\begin{aligned}
&(1) \quad x_{\max}^{(1)} = \frac{m_1 + m_M + m_2}{3}, \\
&(2) \quad x_{\max}^{(2)} = \frac{m_1 + 2m_M + m_2}{4}, \\
&(3) \quad x_{\max}^{(3)} = \frac{m_1 + 4m_M + m_2}{6}.
\end{aligned} \tag{3.16}$$

Contrary to (3.15), the values (3.16) take into consideration the contribution of m_1 and m_2 but give different weight to m_M.

If the triangular number $\mathbf{A}_{ave}$ is close to a central triangular number (see Fig. 1.18 (a)) meaning that m_M is almost in the middle of $[m_1, m_2]$, then (3.15) gives a good crisp value $x_{\max} = m_M$. Then the three average formulas (1)–(3) in (3.16) also produce numbers (maximizing values) close to m_M hence there is no need to be used. Usually in applications the triangular average numbers appear to be in central form. However, the experts dealing with a given situation have to use their judgement when selecting a maximizing value.

The defuzzification procedure is presented as a block diagram in Fig. 3.4.

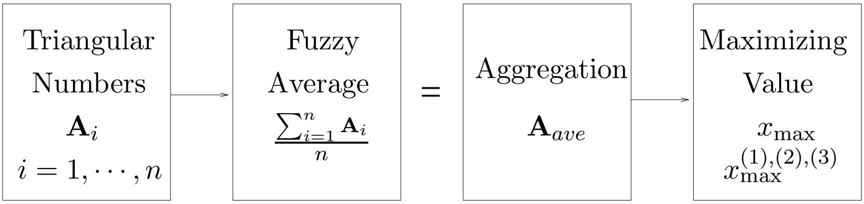

Fig. 3.4. Defuzzification of fuzzy average $\mathbf{A}_{ave} = (m_1, m_2, m_3)$.

For the defuzzification of $\mathbf{A}_{ave}^w = (m_1^w, m_M^w, m_2^w)$ formulas (3.15) and (3.16) remains valid provided m_1^w, m_M^w, m_2^w are substituted for m_1, m_M, m_2 correspondingly.

The defuzzification of the trapezoidal average $\mathbf{A}_{ave} = (m_1, m_{M_1}, m_{M_2}, m_2)$ can be performed by an extension of (3.15) and (3.16) using instead of m_M the midpoint of the flat segment $m_{M_1} m_{M_2}$ at maximum level $\alpha = 1$. The maximizing values are as follows:

$$x_{\max} = \frac{m_{M_1} + m_{M_2}}{2}, \tag{3.17}$$

and

$$\begin{aligned} &(1) \quad x_{\max}^{(1)} = \frac{m_1 + \frac{m_{M_1} + m_{M_2}}{2} + m_2}{3}, \\ &(2) \quad x_{\max}^{(2)} = \frac{m_1 + m_{M_1} + m_{M_2} + m_2}{4}, \\ &(3) \quad x_{\max}^{(3)} = \frac{m_1 + 2(m_{M_1} + m_{M_2}) + m_2}{6}. \end{aligned} \tag{3.18}$$

For the defuzzification of $\mathbf{A}_{ave}^w = (m_1^w, m_{M_1}^w, m_{M_2}^w, m_2^w)$ formulas (3.17) and (3.18) hold but $m_1^w, m_{M_1}^w, m_{M_2}^w, m_2^w$ have to be substituted for $m_1, m_{M_1}, m_{M_2}, m_2$.

Similar block diagrams like that on Fig. 3.4. can be constructed to illustrate defuzzification for the fuzzy averages (3.12)–(3.14).

3.4 Fuzzy Delphi Method for Forecasting

Fuzzy Delphi method is a generalization of the classical method for long range forecasting in management science known as *Delphi method.*

It was developed in the sixties by the Rand Corporation at Santa Monica, California. The name comes from the ancient Greek oracles of Delphi who were famous for forecasting the future.

The essence of Delphi method can be described as follows:

(i) Experts with high qualification regarding a subject are requested to give their opinion separately and independently of each other about the realization dates of a certain event, say in science, technology, or business. They may be asked to forecast the general state of the market, economy, technological advances, etc.

(ii) The data which have subjective character are analyzed statistically by finding their average (see (3.1)) and the results are communicated to the experts.

(iii) The experts review the results and provide new estimates which are analyzed statistically and sent again to the experts for estimation.

(iv) This process could be repeated again and again until the outcome converges to a reasonable solution from the point of view of a manager or a governing body. Usually two or three repetitions are sufficient.

However, long range forecasting problems involve imprecise and incomplete data information. Also the decisions made by the experts rely on their individual competence and are subjective. Therefore it is more appropriate the data to be presented by fuzzy numbers instead of crisp numbers. Especially triangular numbers are very suitable for that purpose since they are constructed easily by specifying three values, the smallest, the largest, and the most plausible (see Section 1.5). Instead of crisp average, the analysis will be based on fuzzy average.

The Fuzzy Delphi method was introduced by Kaufman and Gupta (1988). It consists of the following steps.

Step 1. Experts $E_i, i = 1, \ldots, n$, are asked to provide the possible realization dates of a certain event in science, technology, or business, namely: the earlist date $a_1^{(i)}$, the most plausible date $a_M^{(i)}$, and the latest date $a_2^{(i)}$. The data given by the experts E_i are presented in the form of triangular numbers

$$\mathbf{A}_i = (a_1^{(i)}, a_M^{(i)}, a_2^{(i)}), \quad i = 1, \dots, n. \tag{3.19}$$

Step 2. First, the average (mean) $\mathbf{A}_{ave} = (m_1, m_M, m_2)$ of all $\mathbf{A}_i$ is computed (see (3.11)).

Then for each expert E_i the *deviation* between $\mathbf{A}_{ave}$ and $\mathbf{A}_i$ is computed. It is a triangular number defined by

$$\begin{aligned} \mathbf{A}_{ave} - \mathbf{A}_i &= (m_1 - a_1^{(i)}, m_M - a_M^{(i)}, m_2 - a_2^{(i)}) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} a_1^{(i)} - a_1^{(i)}, \frac{1}{n}\sum_{i=1}^{n} a_M^{(i)} - a_M^{(i)}, \frac{1}{n}\sum_{i=1}^{n} a_2^{(i)} - a_2^{(i)} \right). \end{aligned} \tag{3.20}$$

The deviation $\mathbf{A}_{ave} - \mathbf{A}_i$ is sent back to the expert E_i for reexamination.
Step 3. Each expert E_i presents a new triangular number

$$\mathbf{B}_i = (b_1^{(i)}, b_M^{(i)}, b_2^{(i)}), \quad i = 1, \dots, n. \tag{3.21}$$

This process starting with *Step 2* is repeated. The triangular average $\mathbf{B}_m$ is calculated according to formula (3.11) with the difference that now $a_1^{(i)}, a_M^{(i)}, a_2^{(i)}$ are substituted correspondingly by $b_1^{(i)}, b_M^{(i)}, b_2^{(i)}$. If necessary, new triangular numbers $\mathbf{C}^{(i)} = (c_1^{(i)}, c_M^{(i)}, c_2^{(i)})$ are generated and their average $\mathbf{C}_m$ is calculated. The process could be repeated again and again until two successive means $\mathbf{A}_{ave}, \mathbf{B}_{ave}, \mathbf{C}_{ave}, \dots$ become reasonably close.
Step 4. At a later time the forecasting may be reexamined by the same process if there is important information available due to new discoveries.

Fuzzy Delphi method is a typical multi-experts forecasting procedure for combining views and opinions.

Case Study 1 *Time Estimation for Technical Realization of an Innovative Product*[3]

A group of 15 computer experts are asked to give estimation using Fuzzy Delphi method for the technical realization of a brand new product, say a cognitive information processing computer. They are ranked equally hence their opinions carry the same weight. The triangular numbers $\mathbf{A}_i, i = 1, \ldots, 15$ (see (3.19)) presented by the experts are shown on Table 3.1.

Table 3.1. Triangular numbers $\mathbf{A}_i$ presented by experts (first request).

E_i	$\mathbf{A}_i$	Earliest date	Most plausible date	Lates date
E_1	$\mathbf{A}_1$	$a_1^{(1)} = 1995$	$a_M^{(1)} = 2003$	$a_2^{(1)} = 2020$
E_2	$\mathbf{A}_2$	$a_1^{(2)} = 1997$	$a_M^{(2)} = 2004$	$a_2^{(2)} = 2010$
E_3	$\mathbf{A}_3$	$a_1^{(3)} = 2000$	$a_M^{(3)} = 2005$	$a_2^{(3)} = 2010$
E_4	$\mathbf{A}_4$	$a_1^{(4)} = 1998$	$a_M^{(4)} = 2003$	$a_2^{(4)} = 2008$
E_5	$\mathbf{A}_5$	$a_1^{(5)} = 2000$	$a_M^{(5)} = 2005$	$a_2^{(5)} = 2015$
E_6	$\mathbf{A}_6$	$a_1^{(6)} = 1995$	$a_M^{(6)} = 2010$	$a_2^{(6)} = 2015$
E_7	$\mathbf{A}_7$	$a_1^{(7)} = 2010$	$a_M^{(7)} = 2018$	$a_2^{(7)} = 2020$
E_8	$\mathbf{A}_8$	$a_1^{(8)} = 1995$	$a_M^{(8)} = 2007$	$a_2^{(8)} = 2013$
E_9	$\mathbf{A}_9$	$a_1^{(9)} = 1995$	$a_M^{(9)} = 2002$	$a_2^{(9)} = 2007$
E_{10}	$\mathbf{A}_{10}$	$a_1^{(10)} = 2008$	$a_M^{(10)} = 2009$	$a_2^{(10)} = 2020$
E_{11}	$\mathbf{A}_{11}$	$a_1^{(11)} = 2010$	$a_M^{(11)} = 2020$	$a_2^{(11)} = 2024$
E_{12}	$\mathbf{A}_{12}$	$a_1^{(12)} = 1996$	$a_M^{(12)} = 2002$	$a_2^{(12)} = 2006$
E_{13}	$\mathbf{A}_{13}$	$a_1^{(13)} = 1998$	$a_M^{(13)} = 2006$	$a_2^{(13)} = 2010$
E_{14}	$\mathbf{A}_{14}$	$a_1^{(14)} = 1997$	$a_M^{(14)} = 2005$	$a_2^{(14)} = 2012$
E_{15}	$\mathbf{A}_{15}$	$a_1^{(15)} = 2002$	$a_M^{(15)} = 2010$	$a_2^{(15)} = 2020$

To find the average $\mathbf{A}_{ave}$ the sums of the numbers in the last three columns are calculated

$$\sum_{i=1}^{15} a_1^{(i)} = 29996, \quad \sum_{i=1}^{15} a_M^{(i)} = 30109, \quad \sum_{i=1}^{15} a_2^{(i)} = 30210$$

and substituted into (3.11) which gives

$$\mathbf{A}_{ave} = (\frac{29996}{15}, \frac{30109}{15}, \frac{30210}{15}) = (1999.7, 2007.3, 2014)$$

or approximately

$$\mathbf{A}_{ave}^a = (2000, 2007, 2014).$$

The deviations (3.20) between $\mathbf{A}^a_{ave}$ and $\mathbf{A}_i$ are presented in Table 3.2.

Table 3.2. Deviation $\mathbf{A}^a_{ave} - \mathbf{A}_i$.

E_i	$m_1 - a_1^{(i)}$	$m_M - a_M^{(i)}$	$m_2 - a_2^{(i)}$
E_1	5	4	−6
E_2	3	3	4
E_3	0	2	4
E_4	2	4	6
E_5	0	2	−1
E_6	5	−3	−1
E_7	−10	−11	−6
E_8	5	0	1
E_9	5	5	7
E_{10}	−8	−2	−6
E_{11}	−10	−13	−10
E_{12}	4	5	8
E_{13}	2	1	4
E_{14}	3	2	2
E_{15}	−2	−3	−6

Table 3.2 shows the divergence of each expert's opinion from the average. A quick glance gives that the experts $E_3, E_5, E_8, E_{13}, E_{14}$ are close to the average while E_7, E_{11} are not.

Since the word *close* is fuzzy a more detailed study requires some clarification. It can be based on the concept of distance d_{ij} between two triangular numbers $\mathbf{A}_i$ and $\mathbf{A}_j$. If all d_{ij} are calculated and recorded in a table (in our case consisting of 15 rows and columns), then we will have a better grasp on how close are various pairs of $\mathbf{A}_i$ and $\mathbf{A}_j$. Here we do not give a formula for calculating the distance d_{ij} (there are several),[4] but refer to Kaufmann and Gupta (1988).

Suppose the manager is not satisfied with the average (2000, 2007, 2014). Then the deviation $(m_1 - a_1^{(i)}, m_M - a_M^{(i)}, m_2 - a_2^{(i)})$ is given to each expert E_i for reconsideration. The experts suggest new triangular numbers $\mathbf{B}_i$ (see (3.21)) presented on Table 3.3.

Table 3.3. Triangular numbers presented by experts (second request).

E_i	$\mathbf{B}_i$	Earliest date	Most plausible date	Lates date
E_1	$\mathbf{B}_1$	$b_1^{(1)} = 1996$	$b_M^{(1)} = 2004$	$b_2^{(1)} = 2018$
E_2	$\mathbf{B}_2$	$b_1^{(2)} = 1997$	$b_M^{(2)} = 2004$	$b_2^{(2)} = 2011$
E_3	$\mathbf{B}_3$	$b_1^{(3)} = 2000$	$b_M^{(3)} = 2005$	$b_2^{(3)} = 2011$
E_4	$\mathbf{B}_4$	$b_1^{(4)} = 1998$	$b_M^{(4)} = 2003$	$b_2^{(4)} = 2010$
E_5	$\mathbf{B}_5$	$b_1^{(5)} = 2000$	$b_M^{(5)} = 2005$	$b_2^{(5)} = 2015$
E_6	$\mathbf{B}_6$	$b_1^{(6)} = 1997$	$b_M^{(6)} = 2009$	$b_2^{(6)} = 2015$
E_7	$\mathbf{B}_7$	$b_1^{(7)} = 2005$	$b_M^{(7)} = 2015$	$b_2^{(7)} = 2016$
E_8	$\mathbf{B}_8$	$b_1^{(8)} = 1996$	$b_M^{(8)} = 2007$	$b_2^{(8)} = 2013$
E_9	$\mathbf{B}_9$	$b_1^{(9)} = 1997$	$b_M^{(9)} = 2004$	$b_2^{(9)} = 2010$
E_{10}	$\mathbf{B}_{10}$	$b_1^{(10)} = 2004$	$b_M^{(10)} = 2009$	$b_2^{(10)} = 2017$
E_{11}	$\mathbf{B}_{11}$	$b_1^{(11)} = 2004$	$b_M^{(11)} = 2015$	$b_2^{(11)} = 2016$
E_{12}	$\mathbf{B}_{12}$	$b_1^{(12)} = 1996$	$b_M^{(12)} = 2004$	$b_2^{(12)} = 2006$
E_{13}	$\mathbf{B}_{13}$	$b_1^{(13)} = 1998$	$b_M^{(13)} = 2006$	$b_2^{(13)} = 2010$
E_{14}	$\mathbf{B}_{14}$	$b_1^{(14)} = 1997$	$b_M^{(14)} = 2004$	$b_2^{(14)} = 2012$
E_{15}	$\mathbf{B}_{15}$	$b_1^{(15)} = 2001$	$b_M^{(15)} = 2009$	$b_2^{(15)} = 2015$

The experts E_5, E_{12}, and E_{13} have not change their first estimate. Other experts, for instance E_2, E_3, E_8, E_{14}, made very small changes.

Using again (3.11), this time to find $\mathbf{B}_{ave}$, gives

$$\mathbf{B}_{ave} = (1999.07, 2006.9, 2013.2)$$

which is approximately $\mathbf{B}_{ave}^a = (1999, 2007, 2013)$.

The manager is satisfied that $\mathbf{A}_{ave}$ and $\mathbf{B}_{ave}$, also $\mathbf{A}_{ave}^a$ and $\mathbf{B}_{ave}^a$, are very close (see Fig. 3.5), stops the fuzzy Delphi process, and accepts the triangular number $\mathbf{B}_{ave}^a$ as a combined conclusion of experts' opinions. The interpretation is that the realization of the invention will occur in the time interval [1999, 2013], the supporting interval of the triangular number $\mathbf{B}_{ave}^a$ which is almost in central form. The most likely year for the realization according to the defuzzification formula (3.15) is 2007. Formulas (3.16) produce numbers close to 2007.

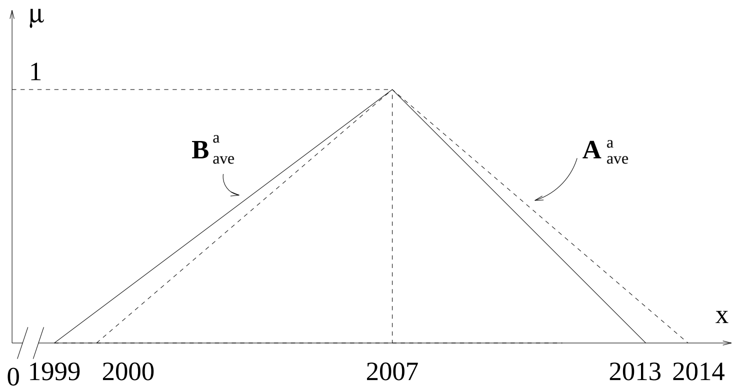

Fig. 3.5. Average triangular numbers $\mathbf{A}^a_{ave}$ and $\mathbf{B}^a_{ave}$.

3.5 Weighted Fuzzy Delphi Method

In business, finance, management, and science, the knowledge, experience, and expertise of some experts is often preferred to the knowledge, experience, and expertise of other experts. This is expressed by weights w_i assigned to the experts (Section 3.3). The experts using Fuzzy Delphi Method (Section 3.4) were considered of equal importance, hence there was no need to introduce weights. Now we consider the case when expert judgements or opinions carry different weights. That leads to *Weighted Fuzzy Delphi Method.*

Assume that to expert $E_i, i = 1, \ldots, n$, is attached a weight $w_i, i = 1, \ldots, n, w_1 + \cdots + w_n = 1$. The four steps in Fuzzy Delphi Method remain valid with some modifications, namely: in *Steps* 2 and 3 the weighted triangular average $\mathbf{A}^w_{ave}$ (see (3.12)) appears instead of the triangular average $\mathbf{A}_{ave}$; in Step 4 similarly $\mathbf{A}^w_{ave}, \mathbf{B}^w_{ave}, \mathbf{C}^w_{ave} \ldots$ take part instead of $\mathbf{A}_{ave}, \mathbf{B}_{ave}, \mathbf{C}_{ave} \ldots$.

Case Study 2 *Weighted Time Estimation for Technical Realization of an Innovative Product*

Consider Case Study 1 where 15 experts present their opinions expressed by triangular numbers $\mathbf{A}_i$ given on Table 3.1. Assume now that the experts $\mathbf{E}_1, \mathbf{E}_3, \mathbf{E}_5, \mathbf{E}_8$, and $\mathbf{E}_{13}$ are ranked higher (weight 0.1) than the rest (weight 0.05); the sum of all weights is one. To facilitate the calculation of the weighted triangular average we construct Table 3.4.

Table 3.4. Experts, weights, and weighted data.

E_i	w_i	$w_i \times a_i^{(i)}$	$w_i \times a_M^{(i)}$	$w_i \times a_2^{(i)}$
E_1	0.1	199.5	200.3	202
E_2	0.05	99.85	100.2	100.5
E_3	0.5	200	200.5	201
E_4	0.05	99.9	100.15	100.4
E_5	0.1	200	200.5	201.5
E_6	0.05	99.75	100.5	100.75
E_7	0.05	100.5	100.9	101
E_8	0.1	199.5	200.7	201.3
E_9	0.05	99.75	100.1	100.35
E_{10}	0.05	100.4	100.45	101
E_{11}	0.05	100.5	101	101.2
E_{12}	0.05	99.8	100.1	100.3
E_{13}	0.1	199.8	200.6	201
E_{14}	0.05	99.85	100.25	100.6
E_{15}	0.05	100.1	100.5	101
Total	1	1999.2	2006.75	2013.9

Substituting the totals from the last row in Table 3.4 into (3.12) gives the weighted triangular average

$$\mathbf{A}^w_{ave} = (1999.2, 2006.75, 2013.9)$$

or approximately $\mathbf{A}^{wa}_{ave} = (1999, 2007, 2014)$. It is almost the same result obtained in Case Study 1. The defuzzification of $\mathbf{A}^{wa}_{ave}$ according to (3.15) produces the year 2007. Formulas (3.16) give close result. If the average $\mathbf{A}^w_{ave}$ is defuzzied instead of $\mathbf{A}^{wa}_{ave}$ and then the maximizing value is rounded up, the same year 2007 is obtained.

□

3.6 Fuzzy PERT for Project Management

Project management is a complicated enterprise involving planning of various activities which have to be performed in the process of development of a new product or technology.

Projects have a specified beginning and end. For convenience they are subdivided into activities which also have specified beginnings and ends. The activities have to be performed in order, some before others, some simultaneously. The time required for completion of each activity has to be *estimated*.

Classical PERT and CPM

Two important classical techniques have been developed to facilitate planning and controlling projects: "Project Evaluation and Review Technique" (PERT) and "Critical Path Method" (CPM).

Table 3.5. Material handling system design, fabrication, and assembly planning data.

	Activity Description	Activities Preceding	Activities Concurrent	Activities Following	Completion time required (days)
A	Mechanical Design	–	–	B, C	35
B	Electrical Design	A	C	D	35
C	Mechanical Fabrication	A	B	E	55
D	Electrical Fabrication	B	C, E	F	35
E	Mechanical Subassembly	C	D	F	50
F	Electrical Installation	D, E	–	G	30
G	Piping Installation	F	–	G	30
H	Start-up, Test, Ship	F	–	–	10

PERT was developed by the U.S.A. Navy while planning the production of Polaris, the nuclear submarine. CPM was developed about the same time by researchers from Remington Rand and DuPont for chemical plant maintenance. There are some similarities between PERT and CPM and often they are used together as one technique.

To illustrate PERT and CPM we present a simplified and modified version of a real project considered by Fogarty and Hoffmann (1983). It is schematically given in Table 3.5. The project, called *Material handling system design*, involves design, fabrication, assembly, and testing. The project is subdivided into eight activities labeled A, B, C, D, E, F, G, H. The completion time for each activity in the last column in Table 3.5 is estimated by managers in charge of activities.

Network planning model

PERT and CPM construct a network planning model from the data in a table. The model corresponding to Table 3.5 is shown in Fig. 3.6. Each activity is represented by a square, rectangle, or circle inside of which is its label and completion time in days.

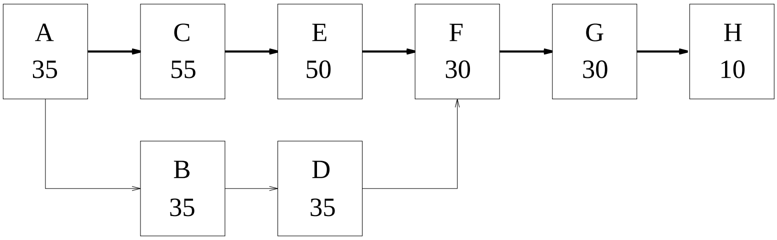

Fig. 3.6. Network planning model for *Material handling system.*

The network planning model gives explicit representation of the sequential relationship between the activities.

Critical path

Critical path is defined as the path of connected-in-sequence activities from beginning to the end of the project that requires the longest completion time. Hence the total time for completion of the project is the time needed to complete the activities on the critical path.

The network planning model helps to determine the critical path. The critical path on Fig. 3.6 is shown by tick arrows connecting activities A, C, E, F, G, and H. The total time for project completion is $35+55+50+30+30+10 = 210$ days. From Fig. 3.6 one can also see that activities B and D are not on the critical path. They may not be completed as planned, but delay should be no more than 35 days. Otherwise activity F on the critical path will be delayed.

Probabilistic PERT

Time estimation or forecasting for activities completion is inherently uncertain. To deal with uncertainty, researchers extended the capability of PERT by employing statistics and probability. PERT requires from experts three estimates for each activity time completion: the *optimistic time* t_1, the time required to complete the activity if everything goes very well; the *most likely time* t_M, the time required to complete the activity if everything goes according to the plan; the *pessimistic time* t_2, the time for completion if there are difficulties or things go wrong. The single time for activity completion is calculated by the weighted average formula

$$t_e = \frac{t_1 + 4t_M + t_2}{6} \tag{3.22}$$

applied for each activity. Formula (3.22) is exactly (3.16) (3) when t is substituted for m. The total time T_e for completion of the project is the time for completion the activities on the critical path. The times calculated from (3.22) for the network planning model on Fig. 3.6 will be close to those presented in the squares and in general will provide a better estimate. The total time T_e (close to 210 days) will be more realistic than 210 days. Further PERT proceeds with calculation of standard deviation for t_e and other probabilistic analysis. We will propose an alternative to the probabilistic PERT which is less complicated.

The three time estimates t_1, t_M, t_2 for each activity come from experts who use their knowledge, experience, and whatever relevant information is available; they are subjective, but not arbitrary. Hence the nature of uncertainties involved in those types of problems is rather fuzzy than probabilistic. PERT does not suggest a technique for finding

t_1, t_M, t_2; only states that they have to be estimated and combined by the statistical weighted average formula (3.22).

Fuzzy PERT for time forecasting

We propose to improve PERT by using Fuzzy Delphi (Section 3.4) for estimating t_1, t_M, t_2 for each activity. Experts represent each time for activity completion by triangular numbers of the type (t_1, t_M, t_2). For each activity the triangular average number is calculated. To find a crisp activity time value we have to use defuzzification (Section 3.3). Simply we may take the maximizing value (formula (3.15)) or resort to the average formulas (3.16)(1)–(3).

The Fuzzy PERT is illustrated in the following case study.

Case Study 3 (Part 1) *Time Forecasting for Project Management of a Material Handling System*

Let us consider the material handling system design on Table 3.5 and Fig. 3.6 and discard the time estimates obtained by the classical PERT. Now each time activity is to be estimated by three experts; some may participate in the estimation time for several activities. The top manager of the project may take part in all group estimates.

The experts are asked to estimate the optimistic, most likely, and pessimistic completion time of activities $A, B, \ldots, H$, expressed as triangular numbers $\mathbf{T}_i^A, \mathbf{T}_i^B, \ldots, \mathbf{T}_i^H, i = 1, 2, 3$.

Suppose that the experts designated to estimate the completion time for activity A produce the results on Table 3.6.

Table 3.6. Estimated completion time for activity A.

Expert	$\mathbf{T}_i^A$	Optimistic time	Most likely time	Pesimistic time
E_1	$\mathbf{T}_1^A$	33	35	38
E_2	$\mathbf{T}_2^A$	33	34	37
E_3	$\mathbf{T}_3^A$	32	36	39
Total	$\sum_{i=1}^3 \mathbf{T}_i^A$	98	105	114

The aggregated experts opinions (see (3.11)) give the average time for completion of A in days

$$\mathbf{T}^A_{ave} = (\frac{98}{3}, \frac{105}{3}, \frac{114}{3}) = (32.67, 35, 38) \approx (33, 35, 38).$$

To find a crisp time for completion we have to defuzzify $\mathbf{T}^A_{ave}$. Observing that $\mathbf{T}_{ave}$ is almost a central triangular number (the midpoint of the interval [32.67, 38] is 35.335, close to 35, we use formula (3.15) which gives $t_{\max} = 35$.

Just for comparison let us apply to $\mathbf{T}^A_{ave}$ the three defuzzification formulas (3.16). We get

$$(1) \quad t^{(1)}_{\max} = \frac{32.67 + 35 + 38}{3} = 35.22,$$

$$(2) \quad t^{(2)}_{\max} = \frac{32.67 + 2(35) + 38}{4} = 35.17,$$

$$(3) \quad t^{(3)}_{\max} = \frac{32.67 + 4(35) + 38}{6} = 35.11,$$

numbers close to 35. Besides, when counting days in those type of projects, it is irrelevant to keep decimals; we round them off and work with full days. Usually decimals appear when working with average formulas.

Similarly the other seven groups of experts can give estimates and construct tables like Table 3.6. We do not give details but assume that the rounded average times $\mathbf{T}^B_{ave}, \ldots, \mathbf{T}^H_{ave}$ are those presented in Table 3.7 ($\mathbf{T}^A_{ave}$ is also included).

Table 3.7. Average times for activities completion.

Activity	Average activity time	Optimistic time t_1	Most likely time t_M	Pesimistic time t_2
A	$\mathbf{T}^A_{ave}$	33	35	38
B	$\mathbf{T}^B_{ave}$	32	35	38
C	$\mathbf{T}^C_{ave}$	51	54	58
D	$\mathbf{T}^D_{ave}$	32	34	36
E	$\mathbf{T}^E_{ave}$	46	50	53
F	$\mathbf{T}^F_{ave}$	27	30	33
G	$\mathbf{T}^G_{ave}$	27	29	32
H	$\mathbf{T}^H_{ave}$	7	10	12

Each triangular number representing the average activity time (the second column in Table 3.7) has to be defuzzified to produce a crisp number expressing the activity completion time. These triangular numbers are almost in central form, hence we can apply formula (3.15) for defuzzification which produces the numbers in the fourth column labeled t_M. The use of formulas (3.16) gives close results.

The defuzzified times can be presented in an improved network planning model (see Fig. 3.7)

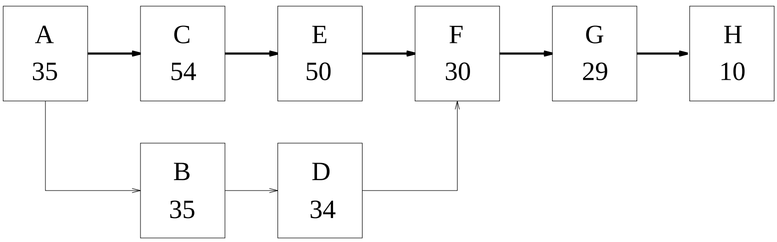

Fig. 3.7. Improved network planning model by using Fuzzy PERT.

The total time for project completion expressed by the triangular number $\mathbf{T}$ is the time for completion the activities on the critical path. Adding the numbers in the three columns in Table 3.7 designated by t_1, t_M, t_2, excluding those belonging to activities B and D, gives

$$\mathbf{T} = \mathbf{T}^A_{ave} + \mathbf{T}^C_{ave} + \mathbf{T}^E_{ave} + \mathbf{T}^F_{ave} + \mathbf{T}^G_{ave} + \mathbf{T}^H_{ave} = (192, 208, 226).$$

Hence the project duration will be between 192 days and 226 days, most likely 208 days. The last number 208 is the result of defuzzification of $\mathbf{T}$ using (3.15). The application of formulas (3.16) for deffuzification generates the crisp numbers $\mathbf{T}^{(1)}_{\max} = 208.67, \mathbf{T}^{(2)}_{\max} = 208.50$, and $\mathbf{T}^{(3)}_{\max} = 208.33$; they are close to 208. As a conclusion the completion time for the project is forecasted to be 208 days.

$\square$

Schedule allocation of resources

Activity time duration and allocation of resources, material and human, are in a close relationship.

It is accepted as common practice that prior to allocation of resources to a project the critical path network should be established.

The forecasting of activity completion times assumes implicitly that the needed resources are available and could be allocated to activities at an efficient rate so that the project proceeds without interruption. In reality various difficulties may arise and complicate the work.

Often management has the option to apply additional resources to reduce the activity completion time. This may increase the cost. Shortening project length may be desirable because of rewards; late completion may be penalized.

PERT helps the analysis of issues like those mentioned above and others concerned with scheduling resources (see for instance, Fogarty and Hoffmann (1983)). For issues requiring estimations, PERT could be combined with Fuzzy Delphi in a fashion similar to activity time forecasting and finding the critical path.

Case Study 3 (Part 2) *Fuzzy PERT for Shortening Project Length*

Following PERT we introduce the notations: t_n—*normal time* for completing an activity as planned, t_c—*crash time* (shorten time) for completing an activity, C_n—*normal cost* for completing an activity, C_c—*crash cost* (increased cost) for completing an activity in crash time. For each activity, t_c, t_n, C_n, and C_c have to be estimated.

We illustrate here Fuzzy PERT for shortening project length on the material handling system discussed in Case Study 3 (Part 1).

To shorten project length means to shorten the time for completion the critical path., i.e. to shorten the total time $T_{\max} = 208$ days. Shortening duration time of activities not on the critical path (B and D, see Fig. 3.6) will not reduce $T_{\max}$. However, some resources allocated to B and D could be reallocated to activities C and D in order to shorten their completion time (internal reallocation). Here we consider shortening activities time on the critical path without internal reallocation of resources.

The normal time t_n for each activity is already estimated; it is the time $t_{\max} = t_M$ shown in Table 3.7, the fourth column.

The crash time t_c, the normal cost C_n, and the crash cost C_c for each activity could be forecasted similarly to the normal time t_n applying

Fuzzy Delphi. The defuzzified values based on formula (3.15) will be denoted by $t_{c\ \max}$, $C_{n\ \max}$, and $C_{c\ \max}$, correspondingly.

Here estimation is presented for the normal cost C_n for activity A; t_c and C_c can be estimated similarly.

Three experts are asked to estimate the normal cost for completion activity A in the form of a triangular number $\mathbf{C}_n = (C_{n1}, C_{nM}, C_{n2})$, where C_{n1} is the lowest cost, C_{nM} is the most likely cost, and C_{n2} is the highest cost. Assume the experts estimates are those in Table 3.8.

Table 3.8. Experts estimate for completion activity A at normal cost C_n.

Expert	Lowest cost C_{n1}	Most likely cost C_{nM}	Highest cost C_{n2}
E_1	18,000	20,000	22,000
E_2	19,500	21,000	22,000
E_3	17,000	19,500	21,000
Total	54,500	60,500	65,000

Using formula (3.11) gives the average normal cost $\mathbf{C}_{n\ ave}^{A}$ for completing activity A,

$$\mathbf{C}_{n\ ave}^{A} = (18,166.67,\ \ 20,166.67,\ \ 21,666.67).$$

Neglecting in $\mathbf{C}_{n\ ave}^{A}$ the decimals and rounding off the last three digits to 000, 500, or 1000, gives

$$\mathbf{C}_{n\ ave}^{A} = (18,000,\ \ 20,000,\ \ 21,500).$$

The defuzzification of $\mathbf{C}_{n\ ave}^{A}$ according to (3.15) produces 20,000 (formulas (3.16) give numbers close to 20,000).

Further, groups of experts forecast t_c, C_n, and C_c for the other activities on the critical path, then defuzzify, and round off as above. Assume that the defuzzified results for the activities on the critical path are those presented in Table 3.9.

To select activities for shortening duration time, PERT uses the notion of *cost slope*. With our notations it is presented as (see Fig. 3.8)

$$k = \textit{cost slope} = \left| \frac{C_{n\ \max} - C_{c\ \max}}{t_{n\ \max} - t_{c\ \max}} \right|. \tag{3.23}$$

Figure 3.8 shows that as normal time $t_{n\ \max}$ decreases approaching the crash time $t_{c\ \max}$, the normal cost $C_{n\ \max}$ increases approaching the crash cost $C_{c\ \max}$.

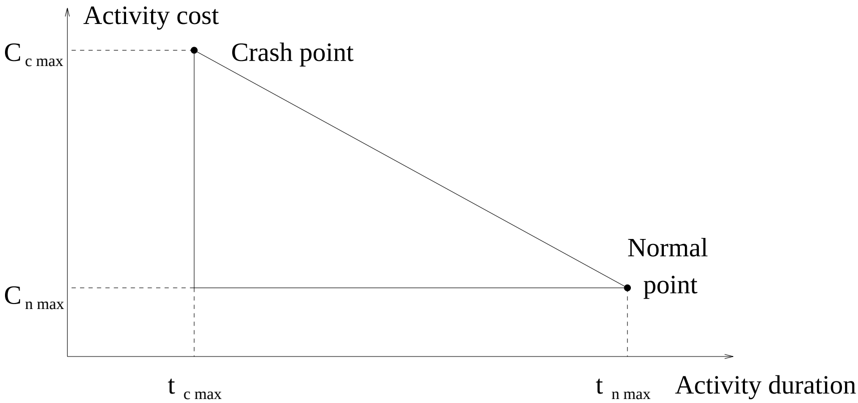

Fig. 3.8. Cost slope for shortening activity time.

Table 3.9. Defuzzified normal and crash times and costs for activities in Material Handling System.

Activity	Normal time $t_{n\ \max}$	Crash time $t_{c\ \max}$	Normal cost $C_{n\ \max}$	Crash cost $C_{c\ \max}$	Cost slope \$ per day
A	35	25	20,000	26,000	600
C	54	30	30,500	40,500	417
E	50	32	28,000	35,000	389
F	30	22	18,500	25,000	813
G	29	20	15,000	19,000	444
H	10	8	7,000	8,000	500

The cost slope coefficient (3.23) calculated for activity A gives

$$k_A = \left|\frac{C_{n\ \max} - C_{c\ \max}}{t_{n\ \max} - t_{c\ \max}}\right| = \left|\frac{20,000 - 26,000}{35 - 25}\right| = \left|\frac{-6000}{10}\right| = 600.$$

The cost slope coefficients for the other activities are calculated similarly. The results are displayed in the last column of Table 3.9.

In general additional resources should be applied first to activities with the smallest cost slope.

The activities in Table 3.9 are ranked in Table 3.10 according to their cost slopes—from the smallest to the largest.

Table 3.10. Ranked activities according to cost slope.

Rank	Activity	Reduced time $t_{n\ \max} - t_{c\ \max}$	Additional cost $C_{c\ \max} - C_{n\ \max}$	Cost slope \$ per day
1	E	18	7,000	389
2	C	24	10,000	417
3	G	9	4,000	444
4	H	2	1,000	500
5	A	10	6,000	600
6	A	8	6,500	813

Assume that the management wants to reduce the length of the project from 208 days to 180 days, a reduction of 28 days. Of the activities on the critical path, activity E ranked first (Table 3.10) has the smallest k, \$ 389 per day. By investing \$ 7,000 the time duration for activity E can be reduced by 18 days, meaning that the project can be reduced by 18 days. A further reduction of 10 days must be found. A good candidate is activity C ranked second on Table 3.10. A 10-day reduction will cost $10 \times 417 = 4,170$ dollars. However, if there are some reasons against shortening the activity time for E or for C, or for both, other options must be examined.

□

3.7 Forecasting Demand

The concept of demand is basic in business and economics. Essentially *demand* is composed of two components expressing: (1) the quantity of a product wanted at a specified price and time; (2) willingness and ability to purchase a product.

Demand for a new product should be forecasted. Forecasting succeeds better when history of demand for a similar product is available.

Unless the product is innovative, even in today's rapidly changing environment, some basic links between the past and the future are present.

The demand for a given inventory item is subdivided into *independent demand* and *dependent demand* (Orlicky, 1975). Demand is independent when it is not related or derived from demand for other items or products. Otherwise demand is called dependent. Independent demand must be forecasted while dependent demand should be determined from the demand of related items.

Example 3.5

Five experts are asked to forecast the annual demand for a new product using Fuzzy Delphi technique which requires use of triangular numbers $\mathbf{A}_i = (a_1^{(i)}, a_M^{(i)}, a_2^{(i)}), i = 1, \ldots, 5$. Here $a_1^{(i)}$ is the smallest number of units to be produced, $a_M^{(i)}$ is the most likely number of units, and $a_2^{(i)}$ is the largest number of units. The experts opinions are shown on Table 3.11.

Table 3.11. Experts estimates for annual demand for a new product.

E_i	$\mathbf{A}_i$	Smallest number $a_1^{(i)}$	Most likely number $a_M^{(i)}$	Largest number $a_2^{(i)}$
E_1	$\mathbf{A}_1$	10,000	12,000	13,000
E_2	$\mathbf{A}_2$	11,000	13,000	15,000
E_3	$\mathbf{A}_3$	10,000	11,000	14,000
E_4	$\mathbf{A}_4$	12,000	13,000	14,000
E_5	$\mathbf{A}_5$	11,000	12,000	13,000
Total		54,000	61,000	69,000

Substituting the total values into (3.11) gives

$$\mathbf{A}_{ave} = \left(\frac{54,000}{5}, \frac{61,000}{5}, \frac{69,000}{5}\right) = (10800, 12200, 13800).$$

The defuzzified $\mathbf{A}_{ave}$ according to (3.15) is 12200. Hence this number can be adopted to represent the annual demand for the new product.

□

3.8 Notes

1. Forecasting in business, finance, and management, regardless of the methodology used, is a controversial subject. A wide range of opinions exist, from claims that forecasting is impossible, to categorical statement that it is a must. Here we present some quotations on the matter by experts and scientists well acquainted with classical techniques for forecasting; there is no evidence that they have knowledge of fuzzy theory.

   "The ability to forecast accurately is central to effective planning strategies. If the forecasts turn out to be wrong, the real cost and opportunity costs ... can be considerable. On the other hand, if they are correct they can provide a great deal of benefit—if the competitors have not followed similar planning strategies"(Makridakis, 1990).

   "To produce an accurate forecast under conditions of stability, the forecaster has merely to conclude that the future will be just like the past. Forecasting may also come out reasonably well if trends change in a way favorable to the organization, for example, if markets grow faster than predicted. Then at least extrapolation does little harm. Typically is overestimation that causes the problems, for example, by projecting a higher demand for a company's products than actually materializes" (Mintzberg, 1994).

   "To claim that forecast is impossible is, of course, a rather extreme way of drawing attention to the frequency with which decision-makers are prone to suffer expensive surprise"(Earl, 1995).

   "The significance of science lies precisely in this: To know in order to foresee .... There is a difference in the degree of foresight and precision achieved in the various sciences." (Leon Trotsky, in *The Age of Permanent Revolution: A Trotsky Anthology*, 1964). The last sentence written in 1940 shows that Trotsky was intuitively close to the concept of fuzziness.

2. Arithmetic operations with fuzzy numbers and in particular with triangular and trapezoidal numbers can be defined by using op-

erations with α-level intervals, level by level (see Kaufmann and Gupta (1985) and G. Bojadziev and M. Bojadziev (1995)).

3. Case Study 1 is based on Kaufmann and Gupta (1988).

4. A simple approximate formula for distance between triangular numbers is given by G. Bojadziev and M. Bojadziev (1995).